BRST-BFV quantization and the Schwinger action principle

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Abstract

We introduce an operator version of the BRST-BFV effective action for arbitrary systems with first-class constraints. Using the Schwinger action principle we calculate the propagators corresponding to: (i) the parametrized non-relativistic free particle, (ii) the relativistic free particle and (iii) the spining relativistic free particle. Our calculation correctly imposes the BRST-invariance at the end-points. The precise

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use of the additional boundary terms required in the description of fermionic variables is also incorporated.
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1   Introduction

A wide variety of interesting theories in physics which range, for example, from the standard model of strong, weak and electromagnetic interactions, to Einstein general relativity and even to more speculative ideas like string theories, can be understood and unified under the generic label of constrained systems. All such gauge theories are characterized by the existence of relations (constraints) among the original phase space variables, together with the appearance of arbitrary functions in the solutions of the equations of motion. The quantum mechanical description of such systems deviates from the standard prescriptions, like the canonical or path integral quantization for unconstrained systems. A systematic procedure for dealing with the quantization of constrained systems was proposed some time ago by Dirac [1] and recently the method has been extended to the BRST-BFV prescription [2] and also to the antifield method of Batalin-Vilkovisky [3], both in the context of the path integral and operator approach to quantization. These methods have been successfully applied to many different problems such as supergravity [4] , topological field theories [5] and superstrings [6], just to mention a few interesting cases.

Nevertheless, in the literature we can find alternative methods of quantization, among which the Schwinger action principle [7] constitutes a very important case. This action principle can be applied to arbitrary quantum systems and starts from an operator formulation of the action from the very beginning. The general validity of this principle has recently motivated its

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application to the case of constrained systems. For example, in reference [8] it is shown that when the Schwinger action principle is applied to a system with only second class constraints, it leads to (anti)commutations relations corresponding precisely to the Dirac bracket prescription. Another application of this action principle has been the calculation of the quantum-mechanical BRST-invariant matrix elements of the evolution operator in the cases of the spinless and the spining relativistic free particle [9], which were previously obtained using the BRST-BFV path integral formulation in Refs. [10],[11]. Unfortunately, the calculation in Ref. [9] makes use of an incorrect (i.e. non-BRST invariant) basis for the physical states at the initial and final times.

The Schwinger action principle can be viewed as a generalization of the Weiss action principle in classical mechanics to the quantum case . The Weiss principle states that if we make an arbitrary variation of the action

$$S = \int_{t'}^{t''} L(\dot{q}, q, t) \, dt, \tag{1}$$

the Euler-Lagrange equations follows from the requirement

$$\delta S = G(t'') - G(t'), \tag{2}$$

where the G's are boundary terms [14] . This means that such variation must not depend upon the trajectory that connects the end-points, thus leading to the Euler-Lagrange equations of motion. The explicit form of the boundary terms depends upon the dynamical variables which are kept fixed at boundaries. For example, if we fix the coordinates q at the boundaries t'' and t', then

$$G(t) = (p\delta q - H\delta t)|_t, \tag{3}$$

where H is the corresponding Hamiltonian function and the notation is $p\delta q = p_i \delta q^i$, with the summation being over all the degrees of freedom of the system.

The notation of deleting the indices will be frequently used in the sequel, in the cases where no confusion arises.

At the quantum level, the Weiss action principle is replaced by the Schwinger action principle which states that the arbitrary variation of the matrix elements of the evolution operator $< a''|U(t'', t')|b' > \equiv \langle a''t''|b't'\rangle$ is given by

$$\delta\langle a''t''|b't'\rangle = i\langle a''t''|\delta\left(\int_{t'}^{t''} L(\dot{\hat{q}}, \hat{q}, t)\, dt\right)|b't'\rangle. \tag{4}$$

Moreover, the variation of the hermitian action operator $\hat{S} = \int_{t'}^{t''} L(\dot{\hat{q}}, \hat{q}, t)dt$ must depend only upon the end point operators and times, in such a way that

$$\delta\left(\int_{t'}^{t''} L(\dot{\hat{q}}, \hat{q}, t)\, dt\right) = G(\hat{A}'', t'') - G(\hat{B}', t'), \tag{5}$$

where $\hat{A}''$ denotes a complete set of commuting operators at the time t'' with corresponding eigenvalues a'' and analogously for the operators $\hat{B}'$ at t'. That is to say, the variation of the propagator is given by the corresponding matrix elements of the variation of a single quantum mechanical operator: the action operator.

A convenient choice for the quantum Lagrangian is the first order form $\hat{L} = \frac{1}{2}\left(\hat{p}\dot{\hat{q}} + \dot{\hat{q}}\hat{p}\right) - H(\hat{q}, \hat{p}, t)$, where $H(\hat{q}, \hat{p}, t)$ is the hermitian Hamiltonian operator constructed in the usual way starting from the definition $\hat{p} = \frac{\partial \hat{L}}{\partial \dot{\hat{q}}}$. The resulting equations of motion are the standard Hamilton equations for the quantum operators

$$\dot{\hat{q}} = \frac{\partial \hat{H}}{\partial \hat{p}}, \qquad \dot{\hat{p}} = -\frac{\partial \hat{H}}{\partial \hat{q}} \tag{6}$$

and one can identify the corresponding end-point generators as

$$G(\hat{q}, \hat{p}) = \hat{p}\delta\hat{q} - \hat{H}\delta t \equiv G_{\delta_{\hat{q}}} + G_{\delta_t}. \tag{7}$$

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At this stage it is also necessary to specify the operator character of the variations $\delta\hat{q}, \delta\hat{p}$ which imply the above results. When dealing with bosonic (fermionic) operators, called operators of the first (second) kind in Schwinger's notation, the corresponding variations satisfy the standard commutation (anticommutation) rules of even (odd) elements in a Grassmann algebra. The standard notation is that bosonic(fermionic) objects have even(odd) Grassmann parity in the underlying Grassmann algebra.

By considering a canonical transformation which interchanges the roles of $\hat{q}$ and $\hat{p}$, it is possible to identify the generator of infinitesimal transformations in $\hat{p}$ as $G_{\delta\hat{p}} = -\hat{q}\delta\hat{p}$. The above expressions for the generators of the corresponding transformations, together with the quantum mechanical interpretation of them as producing infinitesimal unitary transformations, leads to the general commutator

$$\left[\hat{A}, \ \hat{G}_{\delta\hat{b}}\right] = i\delta_{\hat{b}}(\hat{A}). \tag{8}$$

From this expression we obtain the basic (anti)commutation relations for the phase space variables, after taking into account the (anti)commutation properties of the parameters associated to the above transformations.

From Eqs. (4) and (5), the final expression for arbitrary variations of the propagator is then given by

$$\delta\langle a''t''|b't'\rangle = i\langle a''t''|G(A'',t'') - G(B',t')|b't'\rangle. \tag{9}$$

In order to use the above expression as a practical computational tool, we must be able to solve the operator Heisenberg equations of motion for the system in terms of the operators A'', B' whose eigenvalues are kept fixed at the end-points. In this way, we will be able to calculate the corresponding matrix elements in (9), which provide a set of partial differential equations

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for the propagator, that must be subsequently integrated. In other words, we need to choose a complete set of (anti)commuting operators at the initial and final times, together with a well defined inner product in the Hilbert space of physical states, in order to specify the quantum numbers at the end-points which, of course, must be compatible with the dynamics.

In this paper we introduce an operator BRST-BFV action for arbitrary systems with first-class constraints, which is inspired in Schwinger action principle. This action is defined with appropriate BRST-invariant boundary conditions. As an application of this quantum action and as an alternative procedure to the standard path-integral approach, we carefully calculate the propagators corresponding to the non-relativistic particle, the relativistic spinless particle and the relativistic spining particle. The corresponding calculations using the BRST-BFV path integral approach can be found in Refs.[10], [11], [12]. The results presented in our work are based on a consistent choice of end-points conditions and thus allows to clarify some incorrect points that arise in Ref. [9].

The paper is organized as follows: section 2 contains our general prescription to construct the quantum BRST-BFV action, from which we subsequently calculate the corresponding propagators using the Schwinger action principle. The next sections, 3, 4 and 5, contain the corresponding calculations for the following particular cases : the parametrized non-relativistic particle, the relativistic free particle and the spining relativistic free particle, respectively.

2 The quantum BRST- BFV action

Since the action principle does not provide a quantum action to start with, we follow the usual procedure of defining the quantum action as a consistent extension of the classical action associated to the problem.

For a system with constraints, one of the most successful prescriptions to construct a classical gauge independent action is the BRST-BFV method [15]. The resulting action has the advantage of being invariant under BRST transformations and since the remaining symmetry is only global, all the variations of the canonical variables are independent.

We start from a classical system described by canonical coordinates q^i, p_i $(i = 1, \ldots, n)$, having only first-class constraints $G_a(p, q)$ $(a = 1, \ldots, m)$, and with a first-class canonical Hamiltonian $H_0(q, p)$

$$G_a(q, p) \approx 0 \tag{10}$$

$$\{G_a, G_b\}_{PB} = C_{ab}{}^c(q, p)G_c, \quad \{G_a, H_0\}_{PB} = D_a{}^c(q, p)G_c. \tag{11}$$

We assume, for simplicity, that all second-class constraints have been eliminated, either by solving them or by transforming them into first-class constraints, adding new variables and using, for example, the Batalin-Tyutin conversional method [16].

Consider the variational principle in the class of paths $q^i(\tau)$, $p_i(\tau)$, $\lambda^a(\tau)$, where $\lambda^a(\tau)$ are Lagrange multipliers associated to the constraints, with prescribed values at the endpoints τ' and τ'',

$$Q_i(q(\tau'),\ p(\tau'), \tau') = Q'_i, \qquad Q_i(q(\tau''),\ p(\tau''), \tau'') = Q''_i, \tag{12}$$

of a complete set of commuting variables $Q_i(q, p, \tau)$

$$\{Q_i, Q_j\}_{PB} = 0, \quad \text{(at equal times)}. \tag{13}$$

The action for this variational principle is

$$S[q^i(\tau), p_i(\tau), \lambda^a(\tau)] = \int_{\tau'}^{\tau''} (\dot{q}^i p_i - H_0 - \lambda^a G_a) d\tau - B(\tau'') + B(\tau'), \quad (14)$$

(for paths obeying (12)), where the phase space function $B(q, p, \tau)$ is such that

$$p_i \delta q^i = -P^i \delta Q_i + \delta B, \quad (15)$$

for fixed τ [17]. Here, the P^i are the momenta canonically conjugated to the Q_j,

$$\{P^i, P^j\}_{PB} = 0, \quad \{Q_i, P^j\}_{PB} = \delta_i^j. \quad (16)$$

We assume that (14) is the final action of the system arising after we have completed the Dirac procedure of generating all possible secondary constraints and after we have eliminated all second-class constraints. This means that we have already enforced the consistency conditions $\dot{G}_a \approx 0$.

In order to construct the BRST-BFV effective action according to Ref. [15], we start from a configuration space where all degrees of freedom, which can have either even or odd Grassmann parity, are real. Also, we choose the Lagrangian to be real and even. If some coordinate θ is fermionic (odd Grassmann parity), the corresponding momentum p_θ is imaginary and odd in such a way that $\dot{\theta} p_\theta$ is real an even. We assume also that all the constraints are real. They can have either odd or even Grassmann parity. In the later case λ^a is imaginary and odd, so that $\lambda^a G_a$ is both real and even. Next we promote the Lagrange multipliers to the status of dynamical variables by introducing their corresponding canonically conjugated momenta π_a and we demand that $\pi_a \approx 0$, in such a way that we have now $2n$ first-class constraints $G_A = (\pi_a, G_a) \approx 0$. The Grassmann parity ϵ of the new variables is such that $\epsilon(\pi_a) = \epsilon(\lambda^a) = \epsilon(G_a) \equiv \epsilon_a$. The next step is to introduce the ghost

variables η^A together with the corresponding anti-ghost variables $\mathcal{P}_A$, in such a way that $\epsilon(\eta^A) = \epsilon(\mathcal{P}_A) = \epsilon_a + 1$. Following the standard convention we consider the splitting

$$\eta^A = (-(i)^{\epsilon_a+1}\mathcal{P}^a, \mathcal{C}^a), \quad \mathcal{P}_A = ((i)^{\epsilon_a+1}\bar{\mathcal{C}}_a, \bar{\mathcal{P}}_a). \tag{17}$$

The classical effective BRST-BFV action turns out to be

$$S_{BRST} = \int_{\tau'}^{\tau''} \left(\dot{q}^i p_i - \lambda^a \dot{\pi}_a + \dot{\bar{\mathcal{C}}}_a \mathcal{P}^a + \dot{\mathcal{C}}^a \bar{\mathcal{P}}_a - H_{BRST} \right) d\tau - [B]_{\tau'}^{\tau''}, \tag{18}$$

where the integral is extended over the paths which obey the boundary conditions

$$Q_i(q(\tau'),\ p(\tau'), \tau') = Q_i', \qquad Q_i(q(\tau''),\ p(\tau''), \tau'') = Q_i''.$$

$$\mathcal{C}^a(\tau') = \mathcal{C}^a(\tau'') = 0, \qquad \bar{\mathcal{C}}_a(\tau') = \bar{\mathcal{C}}_a(\tau'') = 0, \tag{19}$$

$$\pi_a(\tau') = \pi_a(\tau'') = 0.$$

In Eq.(18), $H_{BRST} = H_c - \{\Psi, \Omega\}_{PB}$, H_c is the canonical Hamiltonian, Ψ is the so called fermionic gauge-fixing term and Ω is the nilpotent BRST-charge, which has odd Grassmann parity and satisfies $\{\Omega,\ \Omega\}_{PB} = 0$. The general form of the BRST charge is $\Omega = -(i)^{\epsilon_a+1}\mathcal{P}^a \pi_a + \mathcal{C}^a G_a +$ "more" , where "more" stands for terms at least quadratic in the ghosts. A systematic algorithm for this construction can be found in Ref. [15]. In all the applications that we will consider in this work, we choose the classical fermionic gauge to be

$$\Psi = \bar{\mathcal{P}}_a \lambda^a, \tag{20}$$

which has odd Grassmann parity.

Let us observe that we can read out directly from the action (18) the classical Poisson brackets for the fundamental variables

$$\{p_i, q^j\}_{PB} = -\delta_i^j = (-)^{\epsilon(q^j)+1}\{q^j, p_i\}_{PB} ,$$

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$$\{\pi_a, \lambda^b\}_{PB} = -\delta_a^b = (-)^{\epsilon_a+1}\{\lambda^b, \pi_a\}_{PB} \; ,$$

$$\{\mathcal{P}^a, \bar{\mathcal{C}}_b\}_{PB} = -\delta_b^a = (-)^{\epsilon_a}\{\bar{\mathcal{C}}_b, \mathcal{P}^a\}_{PB} \; ,$$

$$\{\bar{\mathcal{P}}_a, \mathcal{C}^b\}_{PB} = -\delta_a^b = (-)^{\epsilon_a}\{\mathcal{C}^b, \bar{\mathcal{P}}_a\}_{PB} \; .$$

The above action (18) has two important properties: (i) all canonical variables involved are unconstrained. This feature is reflected in the choice of the associated measure in the path integral formulation of the method, as the corresponding Liouville measure. (ii) the remaining symmetry of the action (18) is a global supersymmetry generated by the BRST charge Ω, which imposes the choice of BRST-invariant end-point conditions.

The classical effective action (18) is our starting point to construct the quantum version of the BRST-BFV method. First, we promote all (imaginary)real phase space variables A, including the ghosts, to (antihermitian)hermitian operators $\hat{A}$. Since the quantum action must be hermitian in order to preserve unitarity, we also adopt the standard replacement for extending real classical products of real variables into hermitian products of hermitian quantum operators

$$(i)^{\epsilon(A)\epsilon(B)}AB \rightarrow \frac{1}{2}(i)^{\epsilon(A)\epsilon(B)}(\hat{A}\hat{B} + (-1)^{\epsilon(A)\epsilon(B)}\hat{B}\hat{A}) \equiv << \hat{A}\hat{B} >> . \quad (21)$$

Let us observe that the operator properties assumed for the variations $\delta\hat{A}, \delta\hat{B}$ precisely guarantee that $\delta << \hat{A}\hat{B} >> = (\delta\hat{A})\hat{B} + \hat{A}(\delta\hat{B})$. In particular, the above prescription has to be applied both to the kinetic term and to the boundary term in the action. The quantum expression for the latter will be discussed in each separate situation and the specific form will be dictated by the classical counterpart. The interplay among the variations of both types of terms will allow the proper identification of the corresponding quantum

generators at the end points, thus providing the basic (anti)commutation relations for the dynamical variables directly from the action principle.

According to the above prescription, the quantum extension of the BRST charge must lead to an hermitian operator $\hat{\Omega}^{\dagger} = \hat{\Omega}$ such that $\{\hat{\Omega}, \hat{\Omega}\} = 2\hat{\Omega}^2 = 0$, with $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denoting the corresponding anticommutator. Both, the canonical Hamiltonian together with the fermionic gauge fixing term are also promoted to the corresponding operators $\hat{H}_c$ (hermitian) and $\hat{\Psi}$ (antihermitian) respectively, while the effective Hamiltonian operator is defined by $\hat{H}_{BRST} = \hat{H}_c + i\{\hat{\Psi}, \hat{\Omega}\}$. Besides, the BRST charge must be conserved i.e. $[\hat{\Omega}, \hat{H}_{BRST}] = 0$, where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ denotes the corresponding commutator.

In this way, the full quantum action turns out to be

$$\hat{S}_{BRST} = \int_{\tau'}^{\tau''} \left(<< \dot{\hat{q}}^i \hat{p}_i - \hat{\lambda}^a \dot{\hat{\pi}}_a + \dot{\hat{\bar{\mathcal{C}}}}_a \hat{\mathcal{P}}^a + \dot{\hat{\mathcal{C}}}^a \hat{\bar{\mathcal{P}}}_a >> -\hat{H}_{BRST} \right) d\tau - [\hat{B}]_{\tau'}^{\tau''}. \tag{22}$$

The basis vectors of the Hilbert space at the initial time τ', $|Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle$, are labeled by the corresponding fixed eigenvalues and satisfy

$$\hat{Q}_i |Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle = Q_i'|Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle \tag{23}$$

$$\hat{\mathcal{C}}^a |Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle = \hat{\bar{\mathcal{C}}}_a |Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle = \hat{\pi}_a |Q_i', \mathcal{C}'^a, \bar{\mathcal{C}}_a', \pi_a'\rangle = 0, \tag{24}$$

according to the classical boundary conditions (19). Analogous expressions are valid for the basis vectors at the final time τ''.

The invariance of the action under quantum BRST transformations is stated in the property $\delta_{\Omega}\hat{S}_{BRST} = i[\hat{\Omega}, \hat{S}_{BRST}]_- = 0$. The BRST invariance of the related transition amplitudes $\langle a''t''|\hat{S}|b't'\rangle$ is guaranteed provided the end point states are also invariant under this transformation, which means that $\hat{\Omega}|b't'\rangle = 0 = \hat{\Omega}|a''t''\rangle$.

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3 The parametrized non-relativistic particle

The classical action for this system is

$$S = \int_{\tau'}^{\tau''} L d\tau = \frac{m}{2} \int_{\tau'}^{\tau''} \frac{\dot{x}^2}{\dot{t}} \, d\tau. \tag{25}$$

Next we define $p_x = \frac{\partial L}{\partial \dot{x}}$, $p_t = \frac{\partial L}{\partial \dot{t}}$ as the momenta canonically conjugated to the coordinates x and t respectively. Here the dot means the derivative with respect to the parameter τ. In this case, the canonical Hamiltonian H_c is zero and the application of the standard Dirac procedure leads to only one (first-class) constraint

$$G = H_0 + p_t \approx 0, \tag{26}$$

where

$$H_0 \equiv \frac{p_x^2}{2m}. \tag{27}$$

Our application of the Schwinger action principle will start from the effective action operator constructed according to the ideas of the previous section. In this case, the subindex a takes just one value, corresponding to the only constraint of the problem . The quantum effective action is taken to be

$$
\begin{aligned}
\hat{S}_{BRST} &= \int_{\tau'}^{\tau''} \left(<< \dot{\hat{x}}\hat{p}_x + \dot{\hat{t}}\hat{p}_t - \hat{\lambda}\dot{\hat{\pi}} + \dot{\hat{\bar{\mathcal{C}}}}\hat{\mathcal{P}} + \dot{\hat{\mathcal{C}}}\hat{\bar{\mathcal{P}}} >> -\hat{H}_{BRST} \right) d\tau \\
&\quad + << \hat{x}'\hat{p}'_x + \hat{t}'\hat{p}'_t >>,
\end{aligned} \tag{28}
$$

where

$$\hat{H}_{BRST} = i\{\hat{\Psi}, \ \hat{\Omega}\}, \quad \hat{\Psi} = \hat{\bar{\mathcal{P}}}\hat{\lambda}, \quad \hat{\Omega} = -i\hat{\mathcal{P}}\hat{\pi} + \hat{\mathcal{C}} \left(\frac{\hat{p}_x^2}{2m} + \hat{p}_t \right). \tag{29}$$

In the sequel, all the canonical variables are considered to be operators and we drop the hat on top of them in order to simplify the notation. The application

of the action principle to the action (28) leads to the Heisenberg equations of motion, written in the general form of Eqs.(6) in terms of the BRST-Hamiltonian, together with the following identification of the generators of transformations at the end-points

$$
\begin{aligned}
\delta \hat{S}_{BRST} &= \left(p''_x \delta x'' + x' \delta p'_x + p''_t \delta t'' + t' \delta p'_t - \lambda'' \delta \pi'' + \lambda' \delta \pi' - \bar{\mathcal{P}}'' \delta \mathcal{C}'' \right. \\
&\left. + \bar{\mathcal{P}}' \delta \mathcal{C}' - \mathcal{P}'' \delta \bar{\mathcal{C}}'' + \mathcal{P}' \delta \bar{\mathcal{C}}' - H''_{BRST} \delta \tau'' + H'_{BRST} \delta \tau' \right), \quad (30)
\end{aligned}
$$

where the superscript $'$ ($''$) denotes the evaluation of the corresponding operator at $\tau = \tau'$ ($\tau = \tau''$) respectively. According to the property (8), the equation (30) implies the following non-zero (anti)commutation relations at equal times

$$
[x, p_x] = [t, p_t] = [\lambda, \pi] = i \qquad \{\bar{\mathcal{C}}, \mathcal{P}\} = \{\bar{\mathcal{P}}, \mathcal{C}\} = -i. \tag{31}
$$

The equation (30) also implies that the eigenvalues which are kept fixed at the end points correspond to the following operators

$$
p_x(\tau'), \quad p_t(\tau'), \quad \pi(\tau'), \quad \mathcal{C}(\tau'), \quad \bar{\mathcal{C}}(\tau'), \tag{32}
$$

$$
x(\tau''), \quad t(\tau''), \quad \pi(\tau''), \quad \mathcal{C}(\tau''), \quad \bar{\mathcal{C}}(\tau''), \tag{33}
$$

which means that we are selecting the following basis for the Hilbert space

$$
\left\{ |p'_x, p'_t, \pi', \mathcal{C}', \bar{\mathcal{C}}', \tau'\rangle \equiv |\tau'\rangle \right\}, \qquad \left\{ \langle x'', t'', \pi'', \mathcal{C}'', \bar{\mathcal{C}}'', \tau''| \equiv \langle \tau''| \right\}, \tag{34}
$$

at the initial and final end-points respectively. The eigenvalues $\pi', \pi'', \mathcal{C}', \mathcal{C}''$, $\bar{\mathcal{C}}', \bar{\mathcal{C}}''$ are taken to be zero, according to Eq.(24). Our notation is A'' (A') for the eigenvalues of the operator $A(\tau'')$ ($A(\tau')$). However, in order to make the notation not too cumbersome, we will denote with the same letter, both the operator and its corresponding eigenvalue in the sequel, expecting that no confusion arises.

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From the (anti)commutation relations (31) we can show that the BRST operator Ω constructed in (29) is hermitian and nilpotent. Also, the BRST-invariance of the above basis (34) can be directly verified. The calculation of the effective Hamiltonian can now be performed, leading to

$$H_{BRST} = i\bar{\mathcal{P}}\mathcal{P} + \lambda G, \tag{35}$$

which is an hermitian operator satisfying $[H_{BRST}, \Omega] = 0$. The equations of motion can be written in the following explicit form

$$\dot{p}_x = 0, \quad \dot{x} - \frac{\lambda p_x}{m} = 0, \quad \dot{p}_t = 0, \quad \dot{t} - \lambda = 0, \quad \dot{\pi} + G = 0, \quad \dot{\lambda} = 0,$$
$$\dot{\mathcal{P}} = 0, \quad \dot{\bar{\mathcal{C}}} - i\bar{\mathcal{P}} = 0, \quad \dot{\bar{\mathcal{P}}} = 0, \quad \dot{\mathcal{C}} + i\mathcal{P} = 0. \tag{36}$$

Next we consider the calculation of the propagator. The first step is to solve the above operator equations. We obtain the general solution

$$p_x = p_x', \quad x(\tau) = x' + \frac{\lambda p_x}{m}(\tau - \tau'), \quad p_t = p_t', \quad t(\tau) = t' + \lambda(\tau - \tau'), \tag{37}$$

$$\pi(\tau) = \pi' - G(\tau - \tau'), \quad \lambda = \lambda', \tag{38}$$

$$\mathcal{P} = \mathcal{P}', \quad \bar{\mathcal{C}}(\tau) = \bar{\mathcal{C}}' + i\bar{\mathcal{P}}(\tau - \tau'), \quad \bar{\mathcal{P}} = \bar{\mathcal{P}}', \quad \mathcal{C}(\tau) = \mathcal{C}' - i\mathcal{P}(\tau - \tau'), \tag{39}$$

where the superscript $'$ denotes the evaluation of the corresponding operator at $\tau = \tau'$, which are used here to denote arbitrary operator integration constants to be further specified according to the boundary conditions (34). A slightly rewritten expression for the variation for the propagator, obtained from (30), is

$$\delta\langle\tau''|\tau'\rangle = i\langle\tau''|p_x'\delta x'' + x'\delta p_x' + p_t'\delta t'' + t'\delta p_t' - \lambda'(\delta\pi'' - \delta\pi')$$
$$-\bar{\mathcal{P}}'(\delta\mathcal{C}'' - \delta\mathcal{C}') - \mathcal{P}'(\delta\bar{\mathcal{C}}'' - \delta\bar{\mathcal{C}}') - H_{BRST}(\delta\tau'' - \delta\tau')|\tau'\rangle. \tag{40}$$

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The next step is to calculate the corresponding matrix elements. After we incorporate the chosen boundary conditions (34) in the above solutions (37)-(39) of the equations of motion, we can write H_{BRST} in terms of the end-points operators $\mathcal{C}', \mathcal{C}'', \bar{\mathcal{C}}', \bar{\mathcal{C}}''$, together with the constant operators λ and G. The result is

$$H_{BRST} = \frac{i}{(\tau'' - \tau')^2} \left(\bar{\mathcal{C}}''\mathcal{C}'' - \bar{\mathcal{C}}''\mathcal{C}' + \mathcal{C}''\bar{\mathcal{C}}' + \bar{\mathcal{C}}'\mathcal{C}' + (\tau'' - \tau') \right) + \lambda'G \quad (41)$$

where the *well-ordering* ($''$ operators to the left and $'$ operators to the right) has been achieved by using the anticommutator

$$\{\bar{\mathcal{C}}', \mathcal{C}''\} = -(\tau'' - \tau'), \quad (42)$$

which is calculated from the solutions (37)-(39), together with the equal-time (anti)commutators (31). The hermiticity of Eq. (41) can be verified explicitly by using again the relation (42).

All the terms whose matrix elements produce eigenvalues that are fixed to zero at boundaries do not contribute to the propagator, as it is the case of the ghosts and anti-ghosts. Furthermore, reparametrization invariance demands that the propagator be independent of the end-point values of the parameter τ. This is guaranteed provided that the matrix elements of H_{BRST} are zero. In order to show this, we need to calculate the matrix elements for $\lambda = \lambda'$. This can be done as follows: multiply from the left the first Eq.(38) by λ and take the appropriate matrix elements on both sides of the resulting equation. Then, use the fact that the eigenvalues of π are fixed to zero at the boundaries, together with the equal-time commutator of λ and π. The result is

$$(\tau'' - \tau')\langle \tau''|\lambda|\tau'\rangle = -\frac{i\langle \tau''|\tau'\rangle}{(p_x'^{\,2}/2m) + p_t'}, \quad (43)$$

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which immediately implies that $\langle \tau''|H_{BRST}|\tau' \rangle = 0$. As usual, we need to complete the rewriting of the variation (40) in *well ordered form* . In our case, this procedure has to be further applied to the operators x', p''_t and t'. Using the corresponding equations of motion we obtain

$$\delta \langle \tau''|\tau' \rangle \;\; = \;\; i\langle \tau''|p'_x \delta x'' + (x'' - \frac{\lambda' p'_x}{m}(\tau'' - \tau'))\delta p'_x + p'_t \delta t''$$
$$+ \;\; (t'' - \lambda'(\tau'' - \tau'))\delta p'_t|\tau' \rangle. \qquad (44)$$

Finally, after substituting the matrix elements of λ, we are able to integrate the resulting system of partial differential equations, obtaining

$$\langle x'', t'', \tau''|p'_x, p'_t, \tau' \rangle = \exp\{ip'_x x'' + ip'_t t''\}/[(p_x'^{\,2}/2m) + p_t'], \qquad (45)$$

which is the correct propagator for the parametrized free particle.

An important point that we want to emphasize is the following : suppose we have constructed a reparametrization invariant version of an arbitrary theory defined through the Hamiltonian H_0, by introducing the parameter τ in complete analogy to the example considered in this section. Under these circumstances, the extended Hamiltonian will be always proportional to the first-class constraint

$$p_t + H_0(q, p) \approx 0, \qquad (46)$$

which arises as a consequence or the imposed reparametrization invariance. The associated quantum condition upon the physical states is that they must be annihilated by such constraint, which means that such states can not depend on the parameter τ and, consequently, the propagator must also be τ independent. In other words, the matrix elements of the extended Hamiltonian between the physical states must be zero. The same argument is valid for the matrix elements of the BRST-Hamiltonian between physical

states, when we consider a non-canonical fermionic gauge fixing $\Psi = \bar{\mathcal{P}}\lambda$, in the BRST approach for a reparametrization-invariant theory. The latter property, which we have explicitly verified in the case of the parametrized non-relativistic free particle, is in contradiction with the results presented in Ref. [9].

4   The relativistic particle

Before considering this problem, let us emphasize two important points which can be directly inferred from the previous example : (i) in the case where the dynamics of the ghost-antighost sector of the theory decouples from the remaining variables, the effective Hamiltonian has the same form as in Eq.(41), except that G is now replaced by the corresponding first-class constraint. (ii) we can always calculate the matrix elements of the Lagrange multiplier associated with the reparametrization-invariance constraint, by imposing the condition that the matrix elements of the BRST-Hamiltonian are zero.

With this ideas in mind we now consider the calculation of the propagator for the relativistic free particle from the point of view of the BRST-BFV operator formulation. We start from the classical action

$$S = \int_{\tau'}^{\tau''} d\tau \frac{1}{2} \left(\frac{1}{\lambda} \dot{x}^\mu \dot{x}_\mu - \lambda m^2 \right), \tag{47}$$

which is reparametrization-invariant provided λ transforms as a Lagrange multiplier. Here we are taking the standard Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. The corresponding first-class constraint is now

$$G = p^\mu p_\mu + m^2 \approx 0. \tag{48}$$

Our starting point in the quantum problem is the operator effective action

$$S_{BRST} = \int_{\tau'}^{\tau''} \left(<< \dot{x}^\mu p_\mu - \lambda \dot{\pi} + \dot{\bar{\mathcal{C}}}\mathcal{P} + \dot{\mathcal{C}}\bar{\mathcal{P}} >> -H_{BRST} \right) d\tau$$

$$+ << x'^{\mu} p'_{\mu} >>, \tag{49}$$

with

$$H_{BRST} = i\{\Psi, \ \Omega\}, \quad \Psi = \bar{\mathcal{P}}\lambda, \quad \Omega = -i\mathcal{P}\pi + \mathcal{C}\left(p^{\mu}p_{\mu} + m^2\right), \tag{50}$$

where the BRST-charge has the same structure as in Eq.(29) except for the explicit form of the constraint G. Here we are dropping the hats over the operators from the very beginning, in order to simplify the notation. Starting from the action principle, in a manner completely analogous to the previous section, we obtain the following non-zero commutation relations

$$[x^{\mu}, p_{\nu}] = i\delta^{\mu}_{\nu}, \qquad [\lambda, \pi] = i, \tag{51}$$

while the ghosts satisfy those anticommutators given in Eq.(31). The (anti) commutator algebra allows for the calculation of the BRST-Hamiltonian

$$H_{BRST} = i\bar{\mathcal{P}}\mathcal{P} + \lambda(p^2 + m^2), \tag{52}$$

together with the explicit form of the equations of motion

$$\dot{p}_{\mu} = 0, \quad \dot{x}^{\mu} - 2\lambda p^{\mu} = 0, \quad \dot{\pi} + G = 0, \quad \dot{\lambda} = 0, \tag{53}$$

$$\dot{\mathcal{P}} = 0, \quad \dot{\bar{\mathcal{C}}} - i\bar{\mathcal{P}} = 0, \quad \dot{\bar{\mathcal{P}}} = 0, \quad \dot{\mathcal{C}} + i\mathcal{P} = 0. \tag{54}$$

The solution of the above equations is

$$p_{\mu} = p_{\mu}', \quad x^{\mu}(\tau) = x^{\mu\prime} + 2\lambda p_{\mu}(\tau - \tau'), \tag{55}$$

$$\pi(\tau) = \pi' - G(\tau - \tau'), \quad \lambda = \lambda', \tag{56}$$

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}', \quad \bar{\mathcal{C}}(\tau) = \bar{\mathcal{C}}' + i\bar{\mathcal{P}}(\tau - \tau'), \quad \mathcal{P} = \mathcal{P}', \quad \mathcal{C}(\tau) = \mathcal{C}' - i\mathcal{P}(\tau - \tau'), \tag{57}$$

where the primed operators denote integrations constants to be determined according to the choice of the end-point conditions. The BRST-invariant

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boundary conditions are chosen in complete analogy with the previous section by fixing the operators

$$p_\mu(\tau'), \quad \pi(\tau'), \quad \mathcal{C}(\tau'), \quad \bar{\mathcal{C}}(\tau'), \tag{58}$$

$$x^\mu(\tau''), \quad \pi(\tau''), \quad \mathcal{C}(\tau''), \quad \bar{\mathcal{C}}(\tau''), \tag{59}$$

at the end-points. This choice implies that the corresponding basis of the Hilbert space are

$$\left\{ |p_\mu', \pi', \mathcal{C}', \bar{\mathcal{C}}', \tau'\rangle \right\}, \qquad \left\{ \langle x^{\mu''}, \pi'', \mathcal{C}'', \bar{\mathcal{C}}'', \tau''| \right\}, \tag{60}$$

respectively. Again, the eigenvalues $\pi', \pi'', \mathcal{C}', \mathcal{C}'', \bar{\mathcal{C}}', \bar{\mathcal{C}}''$ are taken to be zero in order to enforce the BRST-invariance. Since the effective Hamiltonian for this theory has the same structure as in Eq.(41), we calculate the matrix elements for λ by demanding a null result for the matrix elements of H_{BRST}. The answer is

$$(\tau'' - \tau')\langle \tau''|\lambda|\tau'\rangle = -i\frac{\langle \tau''|\tau'\rangle}{p'^{\,2} + m^2}, \tag{61}$$

which is analogous to that of Eq.(43). Next we calculate the propagator. Its variation is given by

$$\delta\langle \tau''|\tau'\rangle = i\langle \tau''|p_\mu' \delta x^{\mu''} + x^{\mu'} \delta p_\mu' - \lambda'(\delta\pi'' - \delta\pi')$$
$$-\bar{\mathcal{P}}'(\delta\mathcal{C}'' - \delta\mathcal{C}') - \mathcal{P}'(\delta\bar{\mathcal{C}}'' - \delta\bar{\mathcal{C}}') - H_{BRST}(\delta\tau'' - \delta\tau')|\tau'\rangle. \tag{62}$$

Using the solutions given in Eqs. (55)-(57) written in terms of the operators fixed at the end-points, we obtain

$$\delta\langle \tau''|\tau'\rangle = i\langle \tau''|p_\mu' \delta x^{\mu''} + (x^{\mu''} - 2\lambda p_\mu'(\tau'' - \tau'))\delta p_\mu'|\tau'\rangle, \tag{63}$$

in a manner completely similar to the previous case. Finally, introducing the matrix elements of λ and integrating with respect to the end point eigenvalues, we get the result

$$\langle x^{\mu''}, \tau''|p_\mu', \tau'\rangle = \exp\{ip_\mu' x^{\mu''}\}/[p'^{\,2} + m^2], \tag{64}$$

which gives the propagator for the free relativistic particle.

5   The spinning relativistic free particle

As our final example we consider the spinning relativistic free particle. To this end let us start from the following classical action

$$S = \int_{\tau'}^{\tau''} d\tau (\dot{x}^\mu p_\mu + \frac{i}{2}(\dot{\theta}^\mu \theta_\mu + \dot{\theta}_5 \theta_5) - N\mathcal{H} - M\mathcal{Q}_0) - \frac{i}{2}\theta(\tau'') \cdot \theta(\tau') - [B]_{\tau'}^{\tau''}, \quad (65)$$

where the variables $x^\mu, p_\mu, N, \mathcal{H}$ are real-even Grassmann-valued, while $\theta^\mu, \theta_5, \mathcal{Q}_0$ are correspondingly real-odd and M is imaginary-odd, in accordance with our general conventions. The first class constraints $\mathcal{H}$ and $\mathcal{Q}_0$ are

$$\mathcal{H} = p^\mu p_\mu + m^2, \quad \mathcal{Q}_0 = p_\mu \theta^\mu + m\theta_5. \quad (66)$$

The explicitly written boundary term $-\frac{i}{2}\theta(\tau'') \cdot \theta(\tau') = -\frac{i}{2}(\theta^\mu(\tau'')\theta_\mu(\tau') + \theta_5(\tau'')\theta_5(\tau'))$ provides the correct end-point conditions for the fermionic coordinates θ^μ, θ_5 leading to the fixing of the following combinations [10]

$$\frac{1}{2}(\theta^\mu(\tau') + \theta^\mu(\tau'')) \equiv \xi^\mu, \quad \frac{1}{2}(\theta_5(\tau') + \theta_5(\tau'')) \equiv \xi_5, \quad (67)$$

which provide unique solutions to the corresponding first-order equations of motion. There could still be additional boundary terms in the action (65), related to the choice of the end points conditions for the remaining variables, which are contained in B. Next we go through the classical BRST formalism. Let us introduce the vector

$$G_A = (\pi_b, G_b) = (\pi_M, \pi_N, \mathcal{Q}_0, \mathcal{H}), \quad b = 1, 2, \quad (68)$$

where the new variables π_M and π_N are the momenta canonically conjugated to the Lagrange multipliers M and N. Here $\epsilon_1 = 1, \epsilon_2 = 0$. The ghosts and

anti-ghosts are taken to be

$$\eta^A = (-\mathcal{P}^1, -i\mathcal{P}^2, \mathcal{C}^1, \mathcal{C}^2), \qquad \mathcal{P}_A = (\bar{\mathcal{C}}_1, i\bar{\mathcal{C}}_2, \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2), \qquad (69)$$

where $(\mathcal{P}^1, \bar{\mathcal{C}}_1), (\mathcal{C}^1, \bar{\mathcal{P}}_1)$ are even canonically-conjugated ghost-antighost variables while $(\mathcal{P}^2, \bar{\mathcal{C}}_2), (\mathcal{C}^2, \bar{\mathcal{P}}_2)$ are correspondingly odd. With these ingredients we now construct the classical BRST charge. The general expression for the case under consideration is

$$\Omega = \eta^A G_A - \frac{1}{2}(-1)^{\epsilon_B} \eta^B \eta^C C_{CB}^A \mathcal{P}_A, \qquad (70)$$

where ϵ_B is the Grassmann parity of the constraint associated with the variable b and C_{CB}^A are the structure functions of the algebra of constraints, which in this case is given by

$$\{\mathcal{Q}_0, \mathcal{Q}_0\}_{PB} = i\mathcal{H}, \qquad \{\mathcal{Q}_0, \mathcal{H}\}_{PB} = 0. \qquad (71)$$

Accordingly, the only structure function different from zero is $C_{11}^2 = i$. Taking this into account and making the required substitutions in Eq.(70) we get

$$\Omega = -\mathcal{P}^1 \pi_M - i\mathcal{P}^2 \pi_N + \mathcal{C}^1 \mathcal{Q}_0 + \mathcal{C}^2 \mathcal{H} + i(\mathcal{C}^1)^2 \bar{\mathcal{P}}_2, \qquad (72)$$

for the classical BRST charge. The theory considered in this section is also reparametrization invariant and thus the canonical Hamiltonian is zero.

Now we promote all dynamical variables to operators with the following reality properties: $x_\mu, p^\mu, N, \pi_N, \bar{\mathcal{P}}_1, \mathcal{C}^1, \mathcal{P}^1, \bar{\mathcal{C}}_1$ are hermitian-even operators, $\mathcal{P}^2, \bar{\mathcal{P}}_2, M$ are antihermitian-odd operators and $\bar{\mathcal{C}}_2, \mathcal{C}^2, \theta^\mu, \theta_5, \pi_M$ are hermitian-odd operators. The quantum effective action that we start from is

$$
\begin{aligned}
S_{BRST} \;=\; & \int_{\tau'}^{\tau''} d\tau \left(<< \dot{x}^\mu p_\mu + \frac{i}{2}(\dot{\theta}^\mu \theta_\mu + \dot{\theta}_5 \theta_5) - N\dot{\pi}_N - M\dot{\pi}_M \right. \\
& \left. + \dot{\bar{\mathcal{C}}}_1 \mathcal{P}^1 + \dot{\mathcal{C}}^1 \bar{\mathcal{P}}_1 + \dot{\bar{\mathcal{C}}}_2 \mathcal{P}^2 + \dot{\mathcal{C}}^2 \bar{\mathcal{P}}_2 >> -H_{BRST} \right) \\
& + << -\frac{i}{2}\theta(\tau'') \cdot \theta(\tau') + x^{\mu\prime} p_\mu{}' >>,
\end{aligned}
\qquad (73)
$$

where

$$H_{BRST} = i\{\Psi, \ \Omega\}, \quad \Psi =<< \bar{\mathcal{P}}_1 M + \bar{\mathcal{P}}_2 N >>,$$

$$\Omega =<< -\mathcal{P}^1 \pi_M - i\mathcal{P}^2 \pi_N + \mathcal{C}^1 \mathcal{Q}_0 + \mathcal{C}^2 \mathcal{H} + i(\mathcal{C}^1)^2 \bar{\mathcal{P}}_2 >> . \tag{74}$$

The (anti)commutation relations arising from the action principle are

$$[x_\mu, p_\nu] = i\eta_{\mu\nu} \quad \{\theta^\mu, \theta^\nu\} = -\eta^{\mu\nu} \quad \{\theta_5, \theta_5\} = -1, \tag{75}$$

$$\{M, \pi_M\} = -i \quad [N, \pi_N] = i, \tag{76}$$

$$[\bar{\mathcal{P}}_1, \mathcal{C}^1] = -[\mathcal{C}^1, \bar{\mathcal{P}}_1] = [\mathcal{P}^1, \bar{\mathcal{C}}_1] = -[\bar{\mathcal{C}}_1, \mathcal{P}^1] = -i, \tag{77}$$

$$\{\bar{\mathcal{P}}_2, \mathcal{C}^2\} = \{\mathcal{C}^2, \bar{\mathcal{P}}_2\} = \{\bar{\mathcal{C}}_2, \mathcal{P}^2\} = \{\mathcal{P}^2, \bar{\mathcal{C}}_2\} = -i. \tag{78}$$

The calculation of the fermionic anticommutators in Eq.(75) is a particular case of the work in Ref.[8]. Using the above results one can directly verify the anticommutator $\{\Omega, \ \Omega\} = 0$ and also we can calculate the BRST-Hamiltonian

$$H_{BRST} = -\bar{\mathcal{P}}_1 \mathcal{P}^1 + M\mathcal{Q}_0 + 2iM\mathcal{C}^1 \bar{\mathcal{P}}_2 + i\bar{\mathcal{P}}_2 \mathcal{P}^2 + N\mathcal{H}, \tag{79}$$

which leads to the following explicit form for the quantum effective action

$$
\begin{aligned}
S_{BRST} \quad = \quad & \int_{\tau'}^{\tau''} d\tau \left(<< \dot{x}^\mu p_\mu + \frac{i}{2}(\dot{\theta}^\mu \theta_\mu + \dot{\theta}_5 \theta_5) - N\dot{\pi}_N - M\dot{\pi}_M \right. \\
& +\dot{\bar{\mathcal{C}}}_1 \mathcal{P}^1 + \dot{\mathcal{C}}^1 \bar{\mathcal{P}}_1 + \dot{\bar{\mathcal{C}}}_2 \mathcal{P}^2 + \dot{\mathcal{C}}^2 \bar{\mathcal{P}}_2 >> +\bar{\mathcal{P}}_1 \mathcal{P}^1 - M\mathcal{Q}_0 \\
& \left. -2iM\mathcal{C}^1 \bar{\mathcal{P}}_2 - i\bar{\mathcal{P}}_2 \mathcal{P}^2 - N\mathcal{H} \right) \\
& << -\frac{i}{2}\theta(\tau'') \cdot \theta(\tau') + x^{\mu'} p_\mu{}' >> . \tag{80}
\end{aligned}
$$

The reality properties of the remaining operators are: $\mathcal{H}, H_{BRST}$ are hermitian-even, $\mathcal{Q}_0, \Omega$ are hermitian-odd, while Ψ is antihermitian-odd. The corresponding equations of motion are

$$\dot{p}^\mu = 0, \quad \dot{x}^\mu - M\theta^\mu - 2Np^\mu = 0, \quad \dot{\theta}^\mu + iMp^\mu = 0, \quad \dot{\theta}_5 + iMm = 0,$$

$$\dot{\pi}_M + \mathcal{Q}_0 + 2i\mathcal{C}^1\bar{\mathcal{P}}_2 = 0, \quad \dot{M} = 0, \quad \dot{\pi}_N + \mathcal{H} = 0, \quad \dot{N} = 0,$$

$$\dot{\mathcal{P}}^1 = 0, \quad \dot{\mathcal{C}}_1 + \bar{\mathcal{P}}_1 = 0, \quad \dot{\mathcal{C}}^1 + \mathcal{P}^1 = 0, \quad \dot{\bar{\mathcal{P}}}_1 + 2iM\bar{\mathcal{P}}_2 = 0,$$

$$\dot{\mathcal{P}}^2 = 0, \quad \dot{\bar{\mathcal{C}}}_2 - i\bar{\mathcal{P}}_2 = 0, \quad \dot{\bar{\mathcal{P}}}_2 = 0, \quad \dot{\mathcal{C}}^2 - 2iM\mathcal{C}^1 + i\mathcal{P}^2 = 0.$$

The general solution of the above system is

$$p_\mu = p_\mu{}' \, , \mathcal{P}^1 = \mathcal{P}^{1'} \, , M = M' \, , N = N' \, , \mathcal{P}^2 = \mathcal{P}^{2'}, \; \bar{\mathcal{P}}_2 = \bar{\mathcal{P}}'_2, \qquad (81)$$

$$x^\mu(\tau) = x'^\mu + (M\xi^\mu + 2Np^\mu)(\tau - \tau'), \qquad (82)$$

$$\theta^\mu(\tau) = -iMp^\mu\tau + \xi^\mu + \frac{i}{2}Mp^\mu(\tau'' + \tau'), \qquad (83)$$

$$\theta_5(\tau) = -iMm\tau + \xi_5 + \frac{i}{2}Mm(\tau'' + \tau'), \qquad (84)$$

$$\mathcal{C}^1(\tau) = \mathcal{C}'^1 - \mathcal{P}^1(\tau - \tau'), \qquad (85)$$

$$\bar{\mathcal{P}}_1(\tau) = \bar{\mathcal{P}}'_1 - 2iM\bar{\mathcal{P}}_2(\tau - \tau'), \qquad (86)$$

$$\bar{\mathcal{C}}_1(\tau) = \bar{\mathcal{C}}'_1 - (\bar{\mathcal{P}}'_1 - iM\bar{\mathcal{P}}_2(\tau - \tau'))(\tau - \tau'), \qquad (87)$$

$$\bar{\mathcal{C}}_2(\tau) = \bar{\mathcal{C}}'_2 + i\bar{\mathcal{P}}_2(\tau - \tau'), \qquad (88)$$

$$\mathcal{C}^2(\tau) = \mathcal{C}'^2 + iM(2\mathcal{C}'^1 - \mathcal{P}^1(\tau - \tau'))(\tau - \tau') - i\mathcal{P}^2(\tau - \tau'), \qquad (89)$$

$$\pi_N(\tau) = \pi'_N - \mathcal{H}(\tau - \tau'), \qquad (90)$$

$$\pi_M(\tau) = \pi'_M - (\mathcal{Q}_0 + i(2\mathcal{C}'^1 - \mathcal{P}^1(\tau - \tau'))\bar{\mathcal{P}}_2)(\tau - \tau'). \qquad (91)$$

The notation is the same as in the previous sections.

The boundary conditions that we take are completely similar to our previous cases. The only novelty that we encounter here is related to the fermionic degrees of freedom described by the θ-variables. In order to clearly elucidate

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this point, let us consider for a moment the contribution of the fermionic degrees of freedom θ^μ to the change of the effective action

$$
\begin{aligned}
\delta_{\theta_\mu} S_{BRST} &= \int_{\tau'}^{\tau''} d\tau (\frac{i}{2}\dot\theta^\mu \delta\theta_\mu + \frac{i}{2}\delta\dot\theta^\mu \theta_\mu - M\delta\theta_\mu p^\mu) \\
&\quad - \frac{i}{2}(\delta\theta^\mu(\tau'')\theta_\mu(\tau') + \theta^\mu(\tau'')\delta\theta_\mu(\tau')) \\
&= \int_{\tau'}^{\tau''} d\tau ((i\dot\theta^\mu - Mp^\mu)\delta\theta_\mu) + \frac{i}{2}\delta(\theta^\mu(\tau') + \theta^\mu(\tau''))(\theta_\mu(\tau'') - \theta_\mu(\tau')).
\end{aligned}
\tag{92}
$$

Substituting the solution of the equations of motion for θ_μ (83), together with the definition (67) of the variable ξ^μ we obtain

$$
\delta_{\theta_\mu} S_{BRST} = \delta\xi^\mu M p_\mu (\tau'' - \tau').
\tag{93}
$$

The same analysis can be applied to θ_5.

The end-point operators are chosen in such a way that the fixed eigenvalues are

$$
p_\mu(\tau') = p_\mu{}', \qquad x^\mu(\tau'') = x^{\mu\prime\prime}
\tag{94}
$$

$$
\frac{1}{2}(\theta^\mu(\tau') + \theta^\mu(\tau'')) = \xi^\mu, \quad \frac{1}{2}(\theta_5(\tau') + \theta_5(\tau'')) = \xi_5,
\tag{95}
$$

$$
\pi_N(\tau') = \pi_N(\tau'') = \mathcal{C}^i(\tau') = \mathcal{C}^i(\tau'') = \bar{\mathcal{C}}_i(\tau') = \bar{\mathcal{C}}_i(\tau'') = 0, \quad i = 1, 2,
\tag{96}
$$

together with the corresponding BRST-invariant basis

$$
\{\langle x^{\mu\prime\prime}, \pi_N{}'', \pi_M{}'', \theta''^\mu(\xi^\mu), \theta''_5(\xi_5), \mathcal{C}^{i\prime\prime}, \bar{\mathcal{C}}_i'', \tau''|\},
$$

$$
\{|p_\mu', \pi_N', \pi_M', \theta'^\mu(\xi^\mu), \theta'_5(\xi_5), \mathcal{C}^{i\prime}, \bar{\mathcal{C}}_i', \tau'\rangle\}.
\tag{97}
$$

Before going to the calculation of the propagator, let us rewrite the effective Hamiltonian (79) in well ordered form. To this end, we use the equations of

motion together with the following (anti)commutation relations at different times

$$[\bar{\mathcal{C}}_1', \mathcal{C}^{1''}] = -i(\tau'' - \tau'), \quad \{\bar{\mathcal{C}}_2', \mathcal{C}^{2''}\} = -(\tau'' - \tau'). \tag{98}$$

The result is

$$
\begin{aligned}
H_{BRST} &= \frac{1}{(\tau'' - \tau')^2}[-(\bar{\mathcal{C}}_1''\mathcal{C}^{1''} - \bar{\mathcal{C}}_1''\mathcal{C}^{1'} - \mathcal{C}^{1''}\bar{\mathcal{C}}_1' + \bar{\mathcal{C}}_1'\mathcal{C}^{1'}) \\
&+ i(\bar{\mathcal{C}}_2''\mathcal{C}^{2''} - \bar{\mathcal{C}}_2''\mathcal{C}^{2'} + \mathcal{C}^{2''}\bar{\mathcal{C}}_2' + \bar{\mathcal{C}}_2'\mathcal{C}^{2'})] \\
&+ M(\xi^\mu p_\mu' + m\xi_5) + N(p'^2 + m^2), \tag{99}
\end{aligned}
$$

Again, it is a direct matter to verify that H_{BRST} is hermitian.

Before going to the calculation of the propagator it is necessary to establish the following results

$$(\tau'' - \tau')\langle\tau''|N|\tau'\rangle\mathcal{H}' = -i\langle\tau''|\tau'\rangle, \quad (\tau'' - \tau')\langle\tau''|M|\tau'\rangle\mathcal{Q}_0' = i\langle\tau''|\tau'\rangle, \tag{100}$$

where $\mathcal{H}' = (p'^2 + m^2)$ and $\mathcal{Q}_0' = \xi^\mu p_\mu' + m\xi_5$. These matrix elements are calculated from Eqs.(90) and (91) respectively and again imply the condition that the matrix elements of the BRST-Hamiltonian between physical states must be zero.

The general variation of the propagator is

$$\delta\langle\tau''|\tau'\rangle = i\langle\tau''|p_\mu'\delta x^{\mu''} + x^{\mu'}\delta p_\mu' + \delta\xi^\mu M p'^\mu(\tau'' - \tau') + \delta\xi_5 M m(\tau'' - \tau')|\tau'\rangle. \tag{101}$$

After substituting the solution of the equations of motion for x'^μ in terms of the boundary operators and after performing the necessary integrations, we obtain the required propagator

$$\langle\tau''|\tau'\rangle = \exp[ip'x'']\frac{(p_\mu'\xi^\mu + m\xi_5)}{(p'^2 + m^2)}. \tag{102}$$

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In conclusion, starting from the quantized version of the BRST-BFV effective action given in Eq.(22), together with the use of the Schwinger action principle and the imposition of correct BRST-invariant boundary conditions, we have obtained the propagators of the parametrized non-relativistic free particle and of the relativistic free particle, in the spinless and spining cases.

6   Acknowledgements

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